FUNCTIONAL ANALYSIS – 2010-2

1 Topological and metric spaces

1.1 Basic Definitions

Definition 1.1 (Topology)**.** Let S be a set. A subset $\mathcal{T}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{T}$. Then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- Let $U, V \in \mathcal{T}$. Then $U \cap V \in \mathcal{T}$.

A set equipped with a topology is called a *topological space*. The elements of $\mathcal{T}$ are called the *open* sets in S. A complement of an open set in S is called a *closed* set.

Definition 1.2. Let S be a topological space and $x \in S$. Then a subset $U \subseteq S$ is called a *neighborhood* of x iff it contains an open set which in turn contains x.

Definition 1.3. Let S be a topological space and U a subset. The *closure* $\overline{U}$ of U is the smallest closed set containing U. The *interior* $\mathring{U}$ of U is the largest open set contained in U.

Definition 1.4 (base)**.** Let $\mathcal{T}$ be a topology. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a *base* of $\mathcal{T}$ iff the elements of $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$. It is called a *subbase* iff the elements of $\mathcal{T}$ are precisely the finite intersections of unions of elements of $\mathcal{B}$.

Proposition 1.5. *Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. $\mathcal{B}$ is the base of a topology on S iff it satisfies all of the following properties:*

- *$\emptyset \in \mathcal{B}$.*
- *For every $x \in S$ there is a set $U \in \mathcal{B}$ such that $x \in U$.*
- *Let $U, V \in \mathcal{B}$. Then there exits a family $\{W_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{B}$ such that $U \cap V = \bigcup_{\alpha\in A} W_\alpha$.*

Proof. **Exercise.** □

Definition 1.6 (Filter)**.** Let S be a set. A subset $\mathcal{F}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *filter* iff it has the following properties:

- $\emptyset \notin \mathcal{F}$ and $S \in \mathcal{F}$.
- Let $U, V \in \mathcal{F}$. Then $U \cap V \in \mathcal{F}$.
- Let $U \in \mathcal{F}$ and $U \subseteq V \subseteq S$. Then $V \in \mathcal{F}$.

Definition 1.7. Let $\mathcal{F}$ be a filter. A subset $\mathcal{B}$ of $\mathcal{F}$ is called a *base* of $\mathcal{F}$ iff every element of $\mathcal{F}$ contains an element of $\mathcal{B}$.

Proposition 1.8. *Let S be a set and $\mathcal{B} \subseteq \mathfrak{P}(S)$. Then $\mathcal{B}$ is the base of a filter on S iff it satisfies the following properties:*

- *$\emptyset \notin \mathcal{B}$ and $\mathcal{B} \neq \emptyset$.*
- *Let $U, V \in \mathcal{B}$. Then there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$.*

Proof. **Exercise.** □

Let S be a topological space and $x \in S$. It is easy to see that the set of neighborhoods of x forms a filter. It is called the *filter of neighborhoods* of x and denoted by $\mathcal{N}_x$. The family of filters of neighborhoods in turn encodes the topology:

Proposition 1.9. *Let S be a topological space and $\{\mathcal{N}_x\}_{x\in S}$ the family of filters of neighborhoods. Then a subset U of S is open iff for every $x \in U$, there is a set $W_x \in \mathcal{N}_x$ such that $W_x \subseteq U$.*

Proof. **Exercise.** □

Proposition 1.10. *Let S be a set and $\{\mathcal{F}_x\}_{x\in S}$ an assignment of a filter to every point in S. Then this family of filters are the filters of neighborhoods of a topology on S iff they satisfy the following properties:*

1. *For all $x \in S$, every element of $\mathcal{F}_x$ contains x.*
2. *For all $x \in S$ and $U \in \mathcal{F}_x$, there exists $W \in \mathcal{F}_x$ such that $U \in \mathcal{F}_y$ for all $y \in W$.*

Proof. If $\{\mathcal{F}_x\}_{x\in S}$ are the filters of neighborhoods of a topology it is clear that the properties are satisfied: 1. Every neighborhood of a point contains the point itself. 2. For a neighborhood U of x take W to be the interior of U. Then W is a neighborhood for each point in W.

Conversely, suppose $\{\mathcal{F}_x\}_{x\in S}$ satisfies Properties 1 and 2. Given x we define an open neighborhood of x to be an element $U \in \mathcal{F}_x$ such that $U \in \mathcal{F}_y$ for all $y \in U$. This definition is not empty since at least S itself is an open neighborhood of every point x in this way. Moreover, for any $y \in U$, by the same definition, U is an open neighborhood of y. Now take $y \notin U$. Then, by Property 1, U is not an open neighborhood of y. Thus, we obtain a good definition of open set: An open set is a set that is an open neighborhood for one (and thus any) of its points. We also declare the empty set to be open.

We proceed to verify the axioms of a topology. Property 1 of Definition 1.1 holds since S is open and we have declared the empty set to be open. Let $\{U_\alpha\}_{\alpha\in I}$ be a family of open sets and consider their union $U = \bigcup_{\alpha\in I} U_\alpha$. Assume U is not empty (otherwise it is trivially open) and pick $x \in U$. Thus, there is $\alpha \in I$ such that $x \in U_\alpha$. But then $U_\alpha \in \mathcal{F}_x$ and also $U \in \mathcal{F}_x$. This is true for any $x \in U$. Hence, U is open. Consider now open sets U and V. Assume the intersection $U \cap V$ to be non-empty (otherwise its openness is trivial) and pick a point x in it. Then $U \in \mathcal{F}_x$ and $V \in \mathcal{F}_x$ and therefore $U \cap V \in \mathcal{F}_x$. The same is true for any point in $U \cap V$, hence it is open.

It remains to show that $\{\mathcal{F}_x\}_{x\in S}$ are the filters of neighborhoods for the topology just defined. It is already clear that any open neighborhood of a point x is contained in $\mathcal{F}_x$. We need to show that every element of $\mathcal{F}_x$ contains an open neighborhood of x. Take $U \in \mathcal{F}_x$. We define W to be the set of points y such that $U \in \mathcal{F}_y$. This cannot be empty as $x \in W$. Moreover, Property 1 implies $W \subseteq U$. Let $y \in W$, then $U \in \mathcal{F}_y$ and we can apply Property 2 to obtain a subset $V \subseteq W$ with $V \in \mathcal{F}_y$. But this implies $W \in \mathcal{F}_y$. Since the same is true for any $y \in W$ we find that W is an open neighborhood of x. This completes the proof. □

Definition 1.11 (Continuity). Let S, T be topological spaces. A map $f : S \to T$ is called *continuous* iff for every open set $U \in T$ the preimage $f^{-1}(U)$ in S is open. We denote the space of continuous maps from S to T by $\mathrm{C}(S, T)$.

Proposition 1.12. *Let S, T be topological spaces and $f : S \to T$ a map. f is continuous iff for every $x \in S$: $f^{-1}(\mathcal{N}_{f(x)}) \subseteq \mathcal{N}_x$.*

Proof. **Exercise.** □

Proposition 1.13. *Let S, T, U be topological spaces, $f \in \mathrm{C}(S, T)$ and $g \in \mathrm{C}(T, U)$. Then, the composition $g \circ f : S \to U$ is continuous.*

Proof. Immediate. □

Definition 1.14 (Induced Topology)**.** Let S be a topological space and U a subset. Consider the topology given on U by the intersection of each open set on S with U. This is called the *induced topology* on U.

Definition 1.15 (Product Topology)**.** Let S be the cartesian product $S = \prod_{\alpha \in I} S_\alpha$ of a family of topological spaces. Consider subsets of S of the form $\prod_{\alpha \in I} U_\alpha$ where finitely many U_α are open sets in S_α and the others coincide with the whole space $U_\alpha = S_\alpha$. These subsets form the base of a topology on S which is called the *product topology*.

Proposition 1.16. *Let S, T, X be topological spaces and $f \in C(S \times T, X)$. Then the map $f_x : T \to X$ defined by $f_x(y) = f(x, y)$ is continuous for every $x \in S$.*

Proof. Fix $x \in S$. Let U be an open set in X. We want to show that $W := f_x^{-1}(U)$ is open. We do this by finding for any $y \in W$ an open neighborhood of y contained in W. If W is empty we are done, hence assume that this is not so. Pick $y \in W$. Then $(x, y) \in f^{-1}(U)$ with $f^{-1}(U)$ open by continuity of f. Since $S \times T$ carries the product topology there must be open sets $V_x \subseteq S$ and $V_y \subseteq T$ with $x \in V_x$, $y \in V_y$ and $V_x \times V_y \subseteq f^{-1}(U)$. But clearly $V_y \subseteq W$ and we are done. $\square$

Definition 1.17. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on the set S. Then, $\mathcal{T}_1$ is called *finer* than $\mathcal{T}_2$ and $\mathcal{T}_2$ is called *coarser* than $\mathcal{T}_1$ iff all open sets of $\mathcal{T}_2$ are also open sets of $\mathcal{T}_1$.

<u>Exercise</u> 1. Let S be the cartesian product $S = \prod_{\alpha \in I} S_\alpha$ of a family of topological spaces. Show that the product topology is the coarsest topology on S that makes all projections $S \to S_\alpha$ continuous.

1.2 Some properties of topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property*.

Definition 1.18 (Hausdorff)**.** Let S be a topological space. Assume that given any two distinct points $x, y \in S$ we can find open sets $U, V \subset S$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Then, S is said to have the *Hausdorff property*. We also say that S is a *Hausdorff space*.

Definition 1.19. Let S be a topological space. S is called *first-countable* iff for each point in S there exists a countable base of its filter of neighborhoods. S is called *second-countable* iff the topology of S admits a countable base.

Definition 1.20. Let S be a topological space and $U, V \subseteq S$ subsets. U is called *dense* in V iff $V \subseteq \overline{U}$.

Definition 1.21 (separable)**.** A topological space is called *separable* iff it contains a countable dense subset.

Proposition 1.22. *A topological space that is second-countable is separable.*

Proof. **Exercise.** □

Definition 1.23 (open cover)**.** Let S be a topological space and $U \subseteq S$ a subset. A family of open sets $\{U_\alpha\}_{\alpha \in A}$ is called an *open cover* of U iff $U \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Proposition 1.24. *Let S be a second-countable topological space and $U \subseteq S$ a subset. Then, every open cover of U contains a countable subcover.*

Proof. **Exercise.** □

Definition 1.25 (compact)**.** Let S be a topological space and $U \subseteq S$ a subset. U is called *compact* iff every open cover of U contains a finite subcover.

Proposition 1.26. *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

Proof. **Exercise.** □

Proposition 1.27. *The image of a compact set under a continuous map is compact.*

Proof. **Exercise.** □

1.3 Sequences and convergence

Definition 1.28 (convergence of sequences)**.** Let $x := \{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in a topological space S. We say that x has an *accumulation point (or limit point)* p iff for every neighborhood U of x we have $x_k \in U$ for infinitely many $k \in \mathbb{N}$. We say that x *converges* to a point p iff for any neighborhood U of p there is a number $n \in \mathbb{N}$ such that for all $k \geq n$: $x_k \in U$.

Proposition 1.29. *Let S be a first-countable topological space and $x = \{x_n\}_{n \in \mathbb{N}}$ a sequence in S with accumulation point p. Then, x has a subsequence that converges to p.*

Proof. By first-countability choose a countable neighborhood base $\{U_n\}_{n\in\mathbb{N}}$ at p. Now consider the family $\{W_n\}_{n\in\mathbb{N}}$ of open neighborhoods $W_n := \bigcap_{k=1}^n U_k$ at p. It is easy to see that this is again a countable neighborhood base at p. Moreover, it has the property that $W_n \subseteq W_m$ if $n \geq m$. Now, Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in W_1$. Recursively, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in W_{k+1}$. This is possible since W_{k+1} contains infinitely many points of x. Let V be a neighborhood of p. There exists some $k \in \mathbb{N}$ such that $U_k \subseteq V$. By construction, then $W_m \subseteq W_k \subseteq U_k$ for all $m \geq k$ and hence $x_{n_m} \in V$ for all $m \geq k$. Thus, the subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to p. □

Definition 1.30 (convergence of filters)**.** A filter $\mathcal{F}$ on a topological space S is said to *converge* to an element $p \in S$ iff every neighborhood of p is contained in $\mathcal{F}$, i.e., $\mathcal{N}_p \subseteq \mathcal{F}$.

Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. We define the filter $\mathcal{F}_x$ associated with this sequence as follows: $\mathcal{F}_x$ contains all the subsets U of S such that U contains all x_n, except possibly finitely many.

Proposition 1.31. *Let $x := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. Then x converges to a point $p \in S$ iff the associated filter $\mathcal{F}_x$ converges to p.*

Proof. **Exercise.** □

Proposition 1.32. *Let S be a topological space and $U \subseteq S$ a subset. Consider the set A_U of filters on S that contain U. Then, the closure $\overline{U}$ of U coincides with the set of points to which some element in A_U converges.*

Proof. If $U = \emptyset$, then A_U is empty and the proof is trivial. Assume the contrary. If $x \in \overline{U}$, then the intersection of U with the filter $\mathcal{N}_x$ of neighborhoods of x generates a filter that contains U and converges to x as desired. If $x \notin \overline{U}$, then there exists a neighborhood V of x such that $U \cap V = \emptyset$. Suppose a filter $\mathcal{F}$ converges to x. Then $\mathcal{F}$ must contain V, hence cannot contain U, i.e., $\mathcal{F} \notin A_U$. □

Definition 1.33. Let S be a topological space and $U \subseteq S$ a subset. Consider the set B_U of sequences of elements of U. Then the set $\overline{U}^s$ consisting of the points to which some element of B_U converges is called the *sequential closure* of U.

Proposition 1.34. *Let S be a topological space and $U \subseteq S$ a subset. Then, $U \subseteq \overline{U}^s \subseteq \overline{U}$. If, moreover, S is first-countable, then $\overline{U}^s = \overline{U}$.*

Proof. **<u>Exercise</u>.** $\square$

Proposition 1.35. *Let S, T be topological spaces, $f \in \mathrm{C}(S, T)$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in S converging to p. Then, the sequence $f\{(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$.*

Proof. **<u>Exercise</u>.** $\square$

Proposition 1.36. *Let S be a Hausdorff topological space, $\mathcal{F}$ a filter on S converging to a point $p \in S$. Then $\mathcal{F}$ does not converge to any other point in S.*

Proof. **<u>Exercise</u>.** $\square$

Corollary 1.37. *Let S be Hausdorff space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in S which converges to a point $p \in S$. Then, $\{x_n\}_{n\in\mathbb{N}}$ does not converge to any other point in S.*

Definition 1.38. Let S be a topological space and $U \subseteq S$ a subset. U is called *limit point compact* iff every sequence in U has an accumulation point. U is called *sequentially compact* iff every sequence in U contains a converging subsequence.

Proposition 1.39. *Sequential compactness implies limit point compactness. In a first-countable space the converse is also true.*

Proof. **<u>Exercise</u>.** $\square$

Proposition 1.40. *A compact space is limit point compact.*

Proof. Consider a sequence x in a compact space S. Suppose x does not have an accumulation point. Then, for each point $p \in S$ we can choose an open neighborhood U_p which contains only finitely many points of x. However, by compactness, S is covered by finitely many of the sets U_p. But their union can only contain a finite number of points of x, a contradiction. $\square$

1.4 Metric spaces

Definition 1.41. Let S be a set and $d : S \times S \to \mathbb{R}_0^+$ a map with the following properties:

- $d(x, y) = d(y, x) \quad \forall x, y \in S$. (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in S$. (triangle inequality)

- $d(x, y) = 0 \implies x = y \quad \forall x, y \in S$. (definiteness)

Then d is called a *metric* on S. S is also called a *metric space.*

Definition 1.42. If in the above definition the third condition is weakened to

- $d(x, x) = 0 \quad \forall x \in S$,

then d is called a *pseudometric* and S a *pseudometric space.*

Definition 1.43. Let S be a pseudometric space, $x \in S$ and $r > 0$. Then the set $B_r(x) := \{y \in S : d(x, y) < r\}$ is called the *open ball* of radius r centered around x in S. The set $\overline{B}_r(x) := \{y \in S : d(x, y) \leq r\}$ is called the *closed ball* of radius r centered around x in S.

Proposition 1.44. *Let S be a pseudometric space. Then, the open balls in S together with the empty set form the basis of a topology on S. This topology is first-countable and such that closed balls are closed. Moreover, the topology is Hausdorff iff S is metric.*

Proof. **Exercise.** □

Definition 1.45. A topological space is called *metrizable* iff there exists a metric such that the open balls given by the metric are a basis of its topology.

Proposition 1.46. *Let S be a set equipped with two metrics d^1 and d^2. Then, the topology generated by d^2 is finer than the topology generated by d^1 iff for all $x \in S$ and $r_1 > 0$ there exists $r_2 > 0$ such that $B^2_{r_2}(x) \subseteq B^1_{r_1}(x)$. In particular, d^1 and d^2 generate the same topology iff the condition holds both ways.*

Proof. **Exercise.** □

Proposition 1.47 (epsilon-delta criterion)**.** *Let S, T be metric spaces and $f : S \to T$ a map. Then, f is continuous iff for every $x \in S$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.*

Proof. **Exercise.** □

1.5 Elementary properties of metric spaces

Proposition 1.48. *Let S be a metric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x converges to $p \in S$ iff for any $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 : d(x_n, p) < \epsilon$.*

Proof. Immediate. □

Definition 1.49. Let S be a metric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x is called a *Cauchy sequence* iff for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0 : d(x_n, x_m) < \epsilon$.

Proposition 1.50. *Any converging sequence in a metric space is a Cauchy sequence.*

Proof. **Exercise.** □

Proposition 1.51. *Suppose x is a Cauchy sequence in a metric space. If p is accumulation point of x then x converges to p.*

Proof. **Exercise.** □

Definition 1.52. Let S be a metric space and $U \subseteq S$ a subset. If every Cauchy sequence in U converges to a point in U then U is called *complete*.

Proposition 1.53. *A complete subset of a metric space is closed. A closed subset of a complete metric space is complete.*

Proof. **Exercise.** □

Definition 1.54 (Totally boundedness)**.** Let S be a metric space. A subset $U \subseteq S$ is called *totally bounded* iff for any $r > 0$ the set U admits a cover by finitely many open balls of radius r.

Proposition 1.55. *A subset of a metric space is compact iff it is complete and totally bounded.*

Proof. We first show that compactness implies totally boundedness and completeness. Let U be a compact subset. Then, for $r > 0$ cover U by open balls of radius r centered at every point of U. Since U is compact, finitely many balls will cover it. Hence, U is totally bounded. Now, consider a Cauchy sequence x in U. Since U is compact x must have an accumulation point $p \in U$ (Proposition 1.40) and hence (Proposition 1.51) converge to p. Thus, U is complete.

We proceed to show that completeness together with totally boundedness imply compactness. Let U be a complete and totally bounded subset. Assume U is not compact and choose a covering $\{U_\alpha\}_{\alpha\in A}$ of U that does not admit a finite subcovering. On the other hand, U is totally bounded and admits a covering by finitely many open balls of radius $1/2$. Hence, there must be at least one such ball B_1 such that $C_1 := B_1 \cap U$ is not covered by finitely many U_α. Choose a point x_1 in C_1. Observe that C_1 itself is totally bounded. Inductively, cover C_n by finitely many open balls of radius $2^{-(n+1)}$. For at least one of those, call it B_{n+1}, $C_{n+1} := B_{n+1} \cap C_n$ is not covered by finitely many U_α. Choose a point x_{n+1} in C_{n+1}. This process yields a Cauchy sequence $x := \{x_k\}_{k\in\mathbb{N}}$. Since U is complete the sequence converges to a point $p \in U$. There must be $\alpha \in A$ such that $p \in U_\alpha$. Since U_α is open there exists $r > 0$ such that $B_r(p) \subseteq U_\alpha$. This implies, $C_n \subseteq U_\alpha$ for all $n \in \mathbb{N}$ such that $2^{-n+1} < r$. However, this is a contradiction to the C_n not being finitely covered. Hence, U must be compact. □

Proposition 1.56. *The notions of compactness, limit point compactness and sequential compactness are equivalent in a metric space.*

Proof. **Exercise.** □

Proposition 1.57. *A totally bounded metric space is second-countable.*

Proof. **Exercise.** □

Proposition 1.58. *The notions of separability and second-countability are equivalent in a metric space.*

Proof. **Exercise.** □

Theorem 1.59 (Baire's Theorem)**.** *Let S be a complete metric space and $\{U_n\}_{n\in\mathbb{N}}$ a sequence of open and dense subsets of S. Then, the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense in S.*

Proof. Set $U := \bigcap_{n\in\mathbb{N}} U_n$. Let V be an arbitrary open set in S. It suffices to show that $V \cap U \neq \emptyset$. To this end we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of S and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive numbers. Choose $x_1 \in U_1 \cap V$ and then $0 < \epsilon_1 \leq 1$ such that $\overline{B_{\epsilon_1}(x_1)} \subseteq U_1 \cap V$. Now, consecutively choose $x_{n+1} \in U_{n+1} \cap B_{\epsilon_n/2}(x_n)$ and $0 < \epsilon_{n+1} < 2^{-n}$ such that $\overline{B_{\epsilon_{n+1}}(x_{n+1})} \subseteq U_{n+1} \cap B_{\epsilon_n}(x_n)$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy since by construction $d(x_n, x_{n+1}) < 2^{-n}$ for all $n \in \mathbb{N}$. So by completeness it converges to some point $x \in S$. Indeed, $x \in \overline{B_{\epsilon_1}(x_1)} \subseteq V$. On the other hand, $x \in \overline{B_{\epsilon_n}(x_n)} \subseteq U_n$ for all $n \in \mathbb{N}$ and hence $x \in U$. This completes the proof. $\square$

Exercise 2. Give an example of a set S, a sequence x in S and two metrics d^1 and d^2 on S that generate the same topology, but such that x is Cauchy with respect to d^1, but not with respect to d^2.

1.6 Completion of metric spaces

Often it is desirable to work with a complete metric space when one is only given a non-complete metric space. To this end one can construct the *completion* of a metric space. This is detailed in the following exercise.

Exercise 3. Let S be a metric space.

- Let $x := \{x_n\}_{n\in\mathbb{N}}$ and $y := \{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences in S. Show that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists.

- Let T be the set of Cauchy sequences in S. Define the function $\tilde{d} : T \times T \to \mathbb{R}^+_0$ by $\tilde{d}(x, y) := \lim_{n\to\infty} d(x_n, y_n)$. Show that $\tilde{d}$ defines a pseudometric on T.

- Show that $a \sim b \iff \tilde{d}(a, b) = 0$ defines an equivalence relation on T.

- Show that $\overline{S} := T/\sim$ is naturally a metric space.

- Show that $\overline{S}$ is complete. [Hint: First show that given a Cauchy sequence x in S and a subsequence x' of x we have $\tilde{d}(x, x') = 0$. That is, $x \sim y$ in T. Use this to show that for any Cauchy sequence x in S an equivalent Cauchy Sequence x' can be constructed which has a specific asymptotic behavior. For example, x' can be made to satisfy $d(x'_n, x'_m) < \frac{1}{\min(m,n)}$. Now a Cauchy sequence $\hat{x} = \{\hat{x}^n\}_{n\in\mathbb{N}}$ in $\overline{S}$ consists of equivalence classes $\hat{x}^n$ of Cauchy sequences in S. Given some representative x^n of $\hat{x}^n$ show that there is another representative x'^n with specific asymptotic behavior. Using such representatives x'^n for all $n \in \mathbb{N}$ show that the equivalence class in $\overline{S}$ of the diagonal sequence $y := \{x'^n_n\}_{n\in\mathbb{N}}$ is a limit of $\hat{x}$.]

- Show that there is a natural isometric embedding (i.e., a map that preserves the metric) $i_S : S \to \overline{S}$. Furthermore, show that this is a bijection iff S is complete.

Definition 1.60. The metric space $\overline{S}$ constructed above is called the *completion* of the metric space S.

Proposition 1.61 (Universal property of completion)**.** *Let S be a metric space, T a complete metric space and $f : S \to T$ an isometric map. Then, there is a unique isometric map $\overline{f} : \overline{S} \to T$ such that $f = \overline{f} \circ i_S$. Furthermore, the closure of $f(S)$ in T is equal to $\overline{f}(\overline{S})$.*

Proof. **Exercise.** $\square$